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A relaxation process including interaction effects of a two-qubit system

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Abstract

A relaxation process of a two-qubit system is studied by means of the quantum master equation. The two qubits interact with each other via the Ising-type coupling, and one of the two qubits is placed under the influence of a thermal reservoir and the other is isolated from an environmental system. It is shown that the interaction between the two qubits significantly affects the relaxation process. The synthetic effect of the interaction between the qubit–qubit interaction and qubit–reservoir interaction induces the entanglement sudden-birth and the violation of the Bell inequality, even if there is no entanglement in an initial state of the two qubits.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Quantum mechanical properties such as coherence and entanglement are essential for understanding physical phenomena [1] and realizing quantum information processing [2, 3] which includes quantum communication and quantum computation. A physical system in the real world is not isolated from its surrounding environment. Since quantum mechanical properties are very fragile under the influence of an environmental system (a thermal reservoir), understanding the decoherence or relaxation of a quantum system is very important in not only quantum physics but also quantum information science. The investigation of the relaxation processes has a long history in nonequilibrium quantum statistical mechanics [4, 5]. They are studied by means of the phenomenological method [6, 7], the stochastic method [8] and the microscopic method [9]. In particular, the quantum master equation derived by the projection operator method [10–12] is very useful for investigating the relaxation process of a quantum system. It has recently been applied for understanding the decoherence in quantum information processing [13–21].

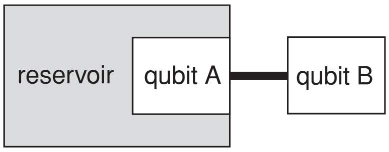

Figure 1. The schematic representation of the interacting two-qubit system. Qubit A interacting with qubit B is embedded in a thermal reservoir. Qubit B is isolated from a thermal reservoir.

When a relevant system consists of interacting subsystems, as pointed out in [4], the effects of the interaction on the relaxation process are essential for deriving correct time evolution and equilibrium state of the relevant system. Since the effects are very difficult to treat, they are neglected in many cases. Recently the decoherence of an interacting two-qubit system has been investigated by the quantum master equations with the damping operators which include the effects of the interaction between the qubits [22–25]. In these works, however, the interaction effects on the relaxation process of the relevant system are not so clear. Therefore, in this paper, in order to find explicitly how an interaction between subsystems affects the decoherence of a relevant system, using the quantum master equation derived by the projection operator method, we will investigate the relaxation process of interacting two qubits under the influence of a thermal reservoir.

This paper is organized as follows. In section 2, we explain a model of two-qubit system, where the two qubits interacts with each other by the Ising coupling and one of the two qubits is placed under the influence of a thermal reservoir. We apply the rotating-wave approximation in the interaction between the qubit and thermal reservoir. In section 3, using the projection operator method, we derive the quantum master equation which describes the reduced dynamics of the two-qubit system. In deriving the quantum master equation, we take into account of the interaction between the two qubits. In section 4, we solve the quantum master equation. In section 5, we investigate the time evolution of the single qubit placed under the influence of the thermal reservoir and show how the interaction between the qubits affects the relaxation process. In section 6, we derive the relaxation process of the two-qubit entanglement. We show that the synthetic effect of the qubit–qubit interaction and the qubit–reservoir interaction induces the entanglement sudden-birth and the violation of the Bell inequality, even though there is no entanglement in an initial state of the two qubits. In section 7, we give concluding remarks.

2. Interacting two qubits in a thermal reservoir

In this section, we explain a model considered in this paper. We suppose that two qubits, A and B, interact with each other through the Ising-type coupling, the Hamiltonian of which is given by

$$\hat{H}_Q = \hbar\omega_A \hat{S}_A^z + \hbar\omega_B \hat{S}_B^z + 2\hbar J \hat{S}_A^z \hat{S}_B^z, \tag{1}$$

where $\hat{S}_{A,B}^{x,y,z}$ is the spin-1/2 operator of the qubit A or B. In the rest of this paper, we assume that the frequencies ω_A and ω_B are positive while the coupling constant J can take positive and negative values. The eigenvalues of the two-qubit Hamiltonian $\hat{H}_Q$ are $E_{00} = (\hbar/2)(\omega_A + \omega_B + J)$, $E_{01} = (\hbar/2)(\omega_A - \omega_B - J)$, $E_{10} = (\hbar/2)(-\omega_A + \omega_B - J)$ and $E_{11} = (\hbar/2)(-\omega_A - \omega_B + J)$ with the corresponding eigenstates $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$, where $\hat{S}_{A,B}^z|0\rangle = (1/2)|0\rangle$ and $\hat{S}_{A,B}^z|1\rangle = -(1/2)|1\rangle$. We suppose that qubit A is placed under the influence of a thermal reservoir while qubit B is isolated from a thermal reservoir. The system that we consider in this paper is depicted in figure 1. Although

Table 1. The reservoir operator $\hat{R}$ in the rotating-wave approximation.

	Qubit B in $\|0\rangle$	Qubit B in $\|1\rangle$
$\omega_A < -J$	$\lambda\hat{R} = \sum_k g_k\hat{a}_k^\dagger$	$\lambda\hat{R} = \sum_k g_k\hat{a}_k$
$\omega_A > \|J\|$	$\lambda\hat{R} = \sum_k g_k\hat{a}_k$	$\lambda\hat{R} = \sum_k g_k\hat{a}_k$
$\omega_A < J$	$\lambda\hat{R} = \sum_k g_k\hat{a}_k$	$\lambda\hat{R} = \sum_k g_k\hat{a}_k^\dagger$

the system is highly asymmetric since the thermal reservoir interacts only with qubit A, such an asymmetric system can be realized by means of the dynamical decoupling method [26, 27] which can be formulated within the framework of the quantum master equation [28–30]. For example, the phase-kick-type dynamical decoupling method [31] can make the interaction between qubit B and the thermal reservoir negligibly small without affecting the qubit–qubit and qubit A–reservoir interactions.

Applying the rotating-wave approximation [32], we have the interaction Hamiltonian $\hat{H}_{\rm int}$ between qubit A and the thermal reservoir

$$\hat{H}_{\rm int} = \hbar\lambda\big(\hat{S}_A^+\hat{R} + \hat{S}_A^-\hat{R}^\dagger\big), \tag{2}$$

where $\hat{S}_A^\pm = \hat{S}_A^x \pm {\rm i}\hat{S}_A^-$ and $\hat{R}$ represents some operator of the thermal reservoir that we determine below. We assume that the thermal reservoir which influences the qubit A consists of independent harmonic oscillators in the thermal equilibrium state. The Hamiltonian of the reservoir is given by $\hat{H}_R = \sum_k \hbar\omega_k\hat{a}_k^\dagger\hat{a}_k$, where $\hat{a}_k$ and $\hat{a}_k^\dagger$ are bosonic annihilation and creation operator, satisfying the commutation relation $\big[\hat{a}_k, \hat{a}_{k'}^\dagger\big] = \delta_{kk'}$. The equilibrium state of the thermal reservoir is $\hat{\rho}_R = (1/Z)\,{\rm e}^{-\hat{H}_R/k_{\rm B}T}$, where Z is the partition function and T is an absolute temperature of the thermal reservoir. Before applying the rotating-wave approximation, the interaction Hamiltonian between qubit A and the reservoir is given by $\hat{H}'_{\rm int} = \sum_k \hbar g_k\big(\hat{S}_A^+ + \hat{S}_A^-\big)\big(\hat{a}_k + \hat{a}_k^\dagger\big)$, where g_k represents the coupling constant. In the interaction picture, we obtain

$$\hat{H}'_{\rm int}(t) = \sum_k g_k\big[\hat{S}_A^+\hat{a}_k\,{\rm e}^{-{\rm i}(\omega_k-\omega_A-2J\hat{S}_B^z)t} + \hat{S}_A^-\hat{a}_k\,{\rm e}^{-{\rm i}(\omega_k+\omega_A+2J\hat{S}_B^z)t} \tag{3}$$

$$+ \hat{S}_A^+\hat{a}_k^\dagger\,{\rm e}^{{\rm i}(\omega_k+\omega_A+2J\hat{S}_B^z)t} + \hat{S}_A^-\hat{a}_k^\dagger\,{\rm e}^{{\rm i}(\omega_k-\omega_A-2J\hat{S}_B^z)t}\big], \tag{4}$$

which clearly shows that the rotating and counter-rotating terms depend on the parameters ω_A and J and the state of qubit B. In fact, we find that the rotating terms are the first term in equation (3) and the second term in equation (4) if qubit B is in the state $|0\rangle$ and $\omega_A + J > 0$ or if qubit B is in the state $|1\rangle$ and $\omega_A - J > 0$ while they are the second term in equation (3) and the first term in equation (4) if the qubit B is in the state $|0\rangle$ and $\omega_A + J < 0$ or if qubit B is in the state $|1\rangle$ and $\omega_A - J < 0$. Thus, the reservoir operator $\hat{R}$ in equation (2) is given by $\lambda\hat{R} = \sum_k g_k\hat{a}_k$ or $\lambda\hat{R} = \sum_k g_k\hat{a}_k^\dagger$, depending on the situation that we consider. The result is summarized in table 1

3. Quantum master equation for the two-qubit system

In order to investigate the reduced dynamics of the interacting two-qubit system, we derive the quantum master equation by means of the projection operator method [9–12]. The time evolution of the density matrix $\hat{W}(t)$ of the total system consisting of the two-qubit system and the thermal reservoir is determined by the Liouville–von Neumann equation,

$$\frac{\partial}{\partial t}\hat{W}(t) = -\frac{\mathrm{i}}{\hbar}[\hat{H}_0 + \hat{H}_{\mathrm{int}}, \hat{W}(t)], \tag{5}$$

where we set the free part $\hat{H}_0 = \hat{H}_Q + \hat{H}_R$ of the total Hamiltonian. In the interaction picture introduced by $\hat{W}^{\mathrm{int}}(t) = \mathrm{e}^{(\mathrm{i}t/\hbar)\hat{H}_0}\hat{W}(t)\,\mathrm{e}^{-(\mathrm{i}t/\hbar)\hat{H}_0}$, we obtain

$$\frac{\partial}{\partial t}\hat{W}^{\mathrm{int}}(t) = -\frac{\mathrm{i}}{\hbar}[\hat{H}_{\mathrm{int}}(t), \hat{W}^{\mathrm{int}}(t)], \tag{6}$$

where $\hat{H}_{\mathrm{int}}(t)$ is given by

$$\hat{H}_{\mathrm{int}}(t) = \hbar\lambda\big[\hat{S}_A^{+}(t)\hat{R}(t) + \hat{S}_A^{-}(t)\hat{R}^{\dagger}(t)\big]. \tag{7}$$

In this equation, we set $\hat{R}(t) = \mathrm{e}^{(\mathrm{i}t/\hbar)\hat{H}_R}\hat{R}\,\mathrm{e}^{-(\mathrm{i}t/\hbar)\hat{H}_R}$ and the spin operator $\hat{S}_A^{\pm}(t)$ is given by

$$\hat{S}_A^{\pm}(t) = \mathrm{e}^{\pm\mathrm{i}(\omega_A + 2J\hat{S}_B^z)t}\,\hat{S}_A^{\pm}. \tag{8}$$

Applying the projection operator method [9–12] to the Liouville–von Neumann equation (6), we can derive the time-convolutionless quantum master equation of the two-qubit system. When the qubit–reservoir interaction is weak, up to the second order with respect to the qubit–reservoir coupling, we obtain

$$\frac{\partial}{\partial t}\hat{W}_Q^{\mathrm{int}}(t) = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t'\,\mathrm{Tr}_R[\hat{H}_{\mathrm{int}}(t), [\hat{H}_{\mathrm{int}}(t'), \hat{W}_Q^{\mathrm{int}}(t)\otimes\hat{\rho}_R]], \tag{9}$$

where $\hat{W}_Q^{\mathrm{int}}(t) = \mathrm{Tr}_R\hat{W}^{\mathrm{int}}(t)$ is the reduced density matrix of the two-qubit system and Tr_R stands for the trace operation over the Hilbert space of the thermal reservoir. Substituting equation (7) into equation (9), we can obtain the time-convolutionless quantum master equation in the Schrödinger picture,

$$\begin{aligned}\frac{\partial}{\partial t}\hat{W}_Q(t) = {} & -\frac{\mathrm{i}}{\hbar}[\hat{H}_Q, \hat{W}_Q(t)] + \phi_{-+}\big(t;\omega_A + 2J\hat{S}_B^z\big)\big[\hat{S}_A^{-}\hat{W}_Q(t), \hat{S}_A^{+}\big] \\ & + \big[\hat{S}_A^{-}, \hat{W}_Q(t)\hat{S}_A^{+}\big]\phi_{-+}^{\dagger}\big(t;\omega_A + 2J\hat{S}_B^z\big) \\ & + \phi_{+-}\big(t;\omega_A + 2J\hat{S}_B^z\big)\big[\hat{S}_A^{+}\hat{W}_Q(t), \hat{S}_A^{-}\big] \\ & + \big[\hat{S}_A^{+}, \hat{W}_Q(t)\hat{S}_A^{-}\big]\phi_{+-}^{\dagger}\big(t;\omega_A + 2J\hat{S}_B^z\big).\end{aligned} \tag{10}$$

In this equation, the functions $\phi_{-+}(t;\omega)$ and $\phi_{+-}(t;\omega)$ are given by

$$\phi_{-+}(t;\omega) = \lambda^2\int_0^t \mathrm{d}t'\,\langle\hat{R}(t')\hat{R}^{\dagger}(0)\rangle\,\mathrm{e}^{\mathrm{i}\omega t'}, \tag{11}$$

$$\phi_{+-}(t;\omega) = \lambda^2\int_0^t \mathrm{d}t'\,\langle\hat{R}^{\dagger}(t')\hat{R}(0)\rangle\,\mathrm{e}^{-\mathrm{i}\omega t'}, \tag{12}$$

where $\langle\cdots\rangle = \mathrm{Tr}[\cdots\hat{\rho}_R]$. In the rest of this paper, we consider the dynamics of the two-qubit system in the Born–Markov approximation [4]. In this approximation, $\phi_{-+}(t;\omega)$ and $\phi_{+-}(t;\omega)$ are replaced with $\phi_{-+}(\infty;\omega) \equiv \phi_{-+}(\omega)$ and $\phi_{+-}(\infty;\omega) \equiv \phi_{+-}(\omega)$. Thus, the quantum master equation of the two-qubit system becomes

$$\frac{\partial}{\partial t}\hat{W}_Q(t) = -\frac{\mathrm{i}}{\hbar}[\hat{H}_Q, \hat{W}_Q(t)] + \hat{\mathcal{L}}_Q^{(0)}\hat{W}_Q(t) + \hat{\mathcal{L}}_Q^{(1)}\hat{W}_Q(t), \tag{13}$$

where the superoperators $\hat{\mathcal{L}}_Q^{(0)}$ and $\hat{\mathcal{L}}_Q^{(1)}$ are given by

$$\begin{aligned}\hat{\mathcal{L}}_Q^{(0)}\hat{W}_Q(t) = {} & \phi_{-+}^{(+)}(\omega_A)\big[\hat{S}_A^{-}\hat{W}_Q(t), \hat{S}_A^{+}\big] + \phi_{-+}^{(+)*}(\omega_A)\big[\hat{S}_A^{-}, \hat{W}_Q(t)\hat{S}_A^{+}\big] \\ & + \phi_{+-}^{(+)}(\omega_A)\big[\hat{S}_A^{+}\hat{W}_Q(t), \hat{S}_A^{-}\big] + \phi_{+-}^{(+)*}(\omega_A)\big[\hat{S}_A^{+}, \hat{W}_Q(t)\hat{S}_A^{-}\big],\end{aligned} \tag{14}$$

and

$$\hat{\mathcal{L}}_Q^{(1)}\hat{W}_Q(t) = \phi_{-+}^{(-)}(\omega_A)\hat{S}_B^z\big[\hat{S}_A^-\hat{W}_Q(t), \hat{S}_A^+\big] + \phi_{-+}^{(-)*}(\omega_A)\big[\hat{S}_A^-, \hat{W}_Q(t)\hat{S}_A^+\big]\hat{S}_B^z + \phi_{+-}^{(-)}(\omega_A)\hat{S}_B^z\big[\hat{S}_A^+\hat{W}_Q(t), \hat{S}_A^-\big] + \phi_{+-}^{(-)*}(\omega_A)\big[\hat{S}_A^+, \hat{W}_Q(t)\hat{S}_A^-\big]\hat{S}_B^z. \tag{15}$$

In these equations, we set

$$\phi_{\mp\pm}^{(+)}(\omega) = \tfrac{1}{2}[\phi_{\mp\pm}(\omega + J) + \phi_{\mp\pm}(\omega - J)], \tag{16}$$

$$\phi_{\mp\pm}^{(-)}(\omega) = \phi_{\mp\pm}(\omega + J) - \phi_{\mp\pm}(\omega - J). \tag{17}$$

The quantum master equation has the two damping parts: one is described by the superoperator $\hat{\mathcal{L}}_Q^{(0)}$ and the other by the superoperator $\hat{\mathcal{L}}_Q^{(1)}$. In the former, the effects of the interaction between the qubits appear only in the coefficients $\phi_{-+}^{(+)}(\omega_A)$ and $\phi_{+-}^{(+)}(\omega_A)$. The operation of each term is the same as that obtained in the absence of the interaction. In the latter, the interaction between the qubits not only modifies the coefficients but also changes the operation of the superoperator since it includes the operator $\hat{S}_B^z$ of the qubit B. Here we remark that the irreversible part of the superoperator $\hat{\mathcal{L}}_Q^{(0)}$ is the diagonal form of the Lindblad operator while that of the superoperator $\hat{\mathcal{L}}_Q^{(1)}$ is the off-diagonal form [9]. Then the quantum master equation (13) is the Lindblad-type equation.

To solve the quantum master equation (13), we consider the matrix elements with respect to the qubit B and we set $\hat{W}_A^{jk}(t) = {}_B\langle j|\hat{W}(t)|k\rangle_B$ $(j,k = 0,1)$ which is an operator of the qubit A. Here $|k\rangle_B$ is the eigenstate of $\hat{S}_B^z$ such that $\hat{S}_B^z|0\rangle_B = \frac{1}{2}|0\rangle_B$ and $\hat{S}_B^z|1\rangle_B = -\frac{1}{2}|1\rangle_B$. Then the reduced density matrix of qubit A is given by $\hat{\rho}_A(t) = \mathrm{Tr}_B\,\hat{W}_Q(t) = \hat{W}_A^{00}(t) + \hat{W}^{11}(t)$. From equation (13), we obtain the equations of motion for the operator $\hat{W}_A^{jk}(t)$,

$$\frac{\partial}{\partial t}\hat{W}_A^{00}(t) = -\mathrm{i}(\omega_A + J)\big[\hat{S}_A^z, \hat{W}_A^{00}(t)\big] + \phi_{-+}(\omega_A + J)\big[\hat{S}_A^-\hat{W}_A^{00}(t), \hat{S}_A^+\big] + \phi_{-+}^*(\omega_A + J)\big[\hat{S}_A^-, \hat{W}_A^{00}(t)\hat{S}_A^+\big] + \phi_{+-}(\omega_A + J)\big[\hat{S}_A^+\hat{W}_A^{00}(t), \hat{S}_A^-\big] + \phi_{+-}^*(\omega_A + J)\big[\hat{S}_A^+, \hat{W}_A^{00}(t)\hat{S}_A^-\big], \tag{18}$$

$$\frac{\partial}{\partial t}\hat{W}_A^{11}(t) = -\mathrm{i}(\omega_A - J)\big[\hat{S}_A^z, \hat{W}_A^{11}(t)\big] + \phi_{-+}(\omega_A - J)\big[\hat{S}_A^-\hat{W}_A^{11}(t), \hat{S}_A^+\big] + \phi_{-+}^*(\omega_A - J)\big[\hat{S}_A^-, \hat{W}_A^{11}(t)\hat{S}_A^+\big] + \phi_{+-}(\omega_A - J)\big[\hat{S}_A^+\hat{W}_A^{11}(t), \hat{S}_A^-\big] + \phi_{+-}^*(\omega_A - J)\big[\hat{S}_A^+, \hat{W}_A^{11}(t)\hat{S}_A^-\big], \tag{19}$$

$$\frac{\partial}{\partial t}\hat{W}_A^{01}(t) = -\mathrm{i}\omega_A\big[\hat{S}_A^z, \hat{W}_A^{01}(t)\big] - \mathrm{i}\omega_B\hat{W}_A^{01}(t) - \mathrm{i}J\big\{\hat{S}_A^z, \hat{W}_A^{01}(t)\big\} + \phi_{-+}(\omega_A + J)\big[\hat{S}_A^-\hat{W}_A^{01}(t), \hat{S}_A^+\big] + \phi_{-+}^*(\omega_A - J)\big[\hat{S}_A^-, \hat{W}_A^{01}(t)\hat{S}_A^+\big] + \phi_{+-}(\omega_A + J)\big[\hat{S}_A^+\hat{W}_A^{01}(t), \hat{S}_A^-\big] + \phi_{+-}^*(\omega_A - J)\big[\hat{S}_A^+, \hat{W}_A^{01}(t)\hat{S}_A^-\big], \tag{20}$$

$$\frac{\partial}{\partial t}\hat{W}_A^{10}(t) = -\mathrm{i}\omega_A\big[\hat{S}_A^z, \hat{W}_A^{10}(t)\big] + \mathrm{i}\omega_B\hat{W}_A^{10}(t) + \mathrm{i}J\big\{\hat{S}_A^z, \hat{W}_A^{10}(t)\big\} + \phi_{-+}(\omega_A - J)\big[\hat{S}_A^-\hat{W}_A^{10}(t), \hat{S}_A^+\big] + \phi_{-+}^*(\omega_A + J)\big[\hat{S}_A^-, \hat{W}_A^{10}(t)\hat{S}_A^+\big] + \phi_{+-}(\omega_A - J)\big[\hat{S}_A^+\hat{W}_A^{10}(t), \hat{S}_A^-\big] + \phi_{+-}^*(\omega_A + J)\big[\hat{S}_A^+, \hat{W}_A^{10}(t)\hat{S}_A^-\big], \tag{21}$$

with $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. Here we introduce the following real parameters which characterize the relaxation process of the two-qubit system:

$$\Delta_{++} = \mathrm{Im}[\phi_{-+}(\omega_A + J) - \phi_{+-}(\omega_A + J)], \tag{22}$$

$$\Delta_{--} = \mathrm{Im}[\phi_{-+}(\omega_A - J) - \phi_{+-}(\omega_A - J)], \tag{23}$$

$$\Delta_{+-} = \mathrm{Im}[\phi_{-+}(\omega_A + J) - \phi_{+-}(\omega_A - J)], \tag{24}$$

$$\Delta_{-+} = \mathrm{Im}[\phi_{-+}(\omega_A - J) - \phi_{+-}(\omega_A + J)], \tag{25}$$

$$\Gamma_{++} = \mathrm{Re}[\phi_{-+}(\omega_A + J) + \phi_{+-}(\omega_A + J)], \tag{26}$$

$$\Gamma_{--} = \mathrm{Re}[\phi_{-+}(\omega_A - J) + \phi_{+-}(\omega_A - J)], \tag{27}$$

$$\Gamma_{+-} = \mathrm{Re}[\phi_{-+}(\omega_A + J) + \phi_{+-}(\omega_A - J)], \tag{28}$$

$$\Gamma_{-+} = \mathrm{Re}[\phi_{-+}(\omega_A - J) + \phi_{+-}(\omega_A + J)] \tag{29}$$

and

$$\sigma_+ = \frac{\mathrm{Re}[\phi_{+-}(\omega_A + J) - \phi_{-+}(\omega_A + J)]}{\mathrm{Re}[\phi_{+-}(\omega_A + J) + \phi_{-+}(\omega_A + J)]}, \tag{30}$$

$$\sigma_- = \frac{\mathrm{Re}[\phi_{+-}(\omega_A - J) - \phi_{-+}(\omega_A - J)]}{\mathrm{Re}[\phi_{+-}(\omega_A - J) + \phi_{-+}(\omega_A - J)]}. \tag{31}$$

Since the reservoir is in the thermal equilibrium state, the Kubo–Martin–Schwinger (KMS) condition [33, 34] is satisfied. For the reservoir operators $\hat{R}$ and $\hat{R}^\dagger$, we obtain the equality $\langle \hat{R}(t)\hat{R}^\dagger(0)\rangle = \langle \hat{R}^\dagger(0)\hat{R}(t + \mathrm{i}\hbar/k_\mathrm{B}T)\rangle$ which yields the detailed balance condition $\mathrm{Re}\,\phi_{-+}(\omega) = \mathrm{e}^{\hbar\omega/k_\mathrm{B}T}\mathrm{Re}\,\phi_{+-}(\omega)$ [9]. Then we find that the parameters σ_+ and σ_- are given by

$$\sigma_\pm = -\tanh\left[\frac{\hbar(\omega \pm J)}{2k_\mathrm{B}T}\right]. \tag{32}$$

If we ignore the effects of the interaction between the two qubits and we set $\hat{\mathcal{L}}_Q^{(1)} = 0$, we have $\sigma_\pm = \sigma_0$ and $\Delta_{jk} = \Delta_0$ and $\Gamma_{jk} = \Gamma_0$ $(j, k = +, -)$ with

$$\sigma_0 = -\tanh\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right), \qquad \mathrm{i}\Delta_0 + \Gamma_0 = \phi_{-+}(\omega) - \phi_{+-}(\omega). \tag{33}$$

The parameters which characterize the relaxation process become independent of the coupling constant J.

4. Time evolution of the two-qubit system

In this section, we solve equations (18)–(21) to find the reduced dynamics of the two-qubit system. First the reduced dynamics of the qubit A is obtained by solving equations (18) and (19):

$$\begin{aligned}\hat{W}_A^{00}(t) = \frac{1}{2}\{&a_1(0)\hat{1} + \mathrm{e}^{-\mathrm{i}(\omega_A+J+\Delta_{++})t-\Gamma_{++}t}a^*(0)\hat{\sigma}^+ + \mathrm{e}^{\mathrm{i}(\omega_A+J+\Delta_{++})t-\Gamma_{++}t}a(0)\hat{\sigma}^- \\ &+ [\mathrm{e}^{-2\Gamma_{++}t}a_z(0) + \sigma_+(1 - \mathrm{e}^{-2\Gamma_{++}t})a_1(0)]\hat{\sigma}_z\},\end{aligned} \tag{34}$$

$$\begin{aligned}\hat{W}_A^{11}(t) = \frac{1}{2}\{&b_1(0)\hat{1} + \mathrm{e}^{-\mathrm{i}(\omega_A-J+\Delta_{--})t-\Gamma_{--}t}b^*(0)\hat{\sigma}^+ + \mathrm{e}^{\mathrm{i}(\omega_A-J+\Delta_{--})t-\Gamma_{--}t}b(0)\hat{\sigma}^- \\ &+ [\mathrm{e}^{-2\Gamma_{--}t}b_z(0) + \sigma_-(1 - \mathrm{e}^{-2\Gamma_{--}t})b_1(0)]\hat{\sigma}_z\},\end{aligned} \tag{35}$$

where the parameters $a_1(0)$, $a(0)$, $a_z(0)$, $b_1(0)$, $b(0)$ and $b_z(0)$ are determined by the initial condition, that is,

$$a(0) = \langle 10|\hat{W}_Q(0)|00\rangle, \tag{36}$$

$$a_1(0) = \langle 00|\hat{W}_Q(0)|00\rangle + \langle 10|\hat{W}_Q(0)|10\rangle, \tag{37}$$

$$a_z(0) = \langle 00|\hat{W}_Q(0)|00\rangle - \langle 10|\hat{W}_Q(0)|10\rangle, \tag{38}$$

$$b(0) = \langle 11|\hat{W}_Q(0)|01\rangle, \tag{39}$$

$$b_1(0) = \langle 01|\hat{W}_Q(0)|01\rangle + \langle 11|\hat{W}_Q(0)|11\rangle, \tag{40}$$

$$b_z(0) = \langle 01|\hat{W}_Q(0)|01\rangle - \langle 11|\hat{W}_Q(0)|11\rangle. \tag{41}$$

In equations (34) and (35), $\hat{\sigma}_{x,y,z}$ is the Pauli matrix and $\hat{\sigma}^{\pm} = (\hat{\sigma}_x + \pm\hat{\sigma}_y)/2$. The the reduced density matrix of the qubit A is given by $\hat{\rho}_A(t) = \hat{W}_A^{00}(t) + \hat{W}_A^{11}(t)$. Since $\mathrm{Tr}\hat{\rho}_A(t) = 1$, the equality $a_1(0) + b_1(0) = 1$ holds.

When we investigate the decay of the two-qubit entanglement, we assume that the two qubits are initially prepared in the X-state [35] which includes the Werner state [36] and the maximally entangled mixed state [37, 38]. The density matrix of the X-state is given by

$$\hat{W}_Q(0) = \begin{pmatrix} a(0) & 0 & 0 & x(0) \\ 0 & b(0) & y(0) & 0 \\ 0 & y^*(0) & c(0) & 0 \\ x^*(0) & 0 & 0 & d(0) \end{pmatrix}, \tag{42}$$

where $a(0)$, $b(0)$, $c(0)$ and $d(0)$ are non-negative and $a(0) + b(0) + c(0) + d(0) = 1$, and $\sqrt{a(0)d(0)} \geqslant |x(0)|$ and $\sqrt{b(0)c(0)} \geqslant |y(0)|$ must be satisfied due to $\hat{W}_Q(0) > 0$ and $\mathrm{Tr}\,\hat{W}_Q(0) = 1$. The quantum master equation (13) preserves the X-state and we obtain the two-qubit state $\hat{W}_Q(t)$ at time t:

$$\hat{W}_Q(t) = \begin{pmatrix} \langle 0|\hat{W}_A^{00}(t)|0\rangle & 0 & 0 & \langle 0|\hat{W}_A^{01}(t)|1\rangle \\ 0 & \langle 0|\hat{W}_A^{11}(t)|0\rangle & \langle 0|\hat{W}_A^{10}(t)|1\rangle & 0 \\ 0 & \langle 1|\hat{W}_A^{01}(t)|0\rangle & \langle 1|\hat{W}_A^{00}(t)|1\rangle & 0 \\ \langle 1|\hat{W}_A^{10}(t)|0\rangle & 0 & 0 & \langle 1|\hat{W}_A^{11}(t)|1\rangle \end{pmatrix}$$

$$= \begin{pmatrix} a(t) & 0 & 0 & x(t) \\ 0 & b(t) & y(t) & 0 \\ 0 & y^*(t) & c(t) & 0 \\ x^*(t) & 0 & 0 & d(t) \end{pmatrix}, \tag{43}$$

with

$$a(t) = \tfrac{1}{2}[1 + \sigma_+ + (1 - \sigma_+)\,\mathrm{e}^{-2\Gamma_{++}t}]a(0) + \tfrac{1}{2}(1 + \sigma_+)[1 - \mathrm{e}^{-2\Gamma_{++}t}]c(0), \tag{44}$$

$$b(t) = \tfrac{1}{2}[1 + \sigma_- + (1 - \sigma_-)\,\mathrm{e}^{-2\Gamma_{--}t}]b(0) + \tfrac{1}{2}(1 + \sigma_-)[1 - \mathrm{e}^{-2\Gamma_{--}t}]d(0), \tag{45}$$

$$c(t) = \tfrac{1}{2}(1 - \sigma_+)[1 - \mathrm{e}^{-2\Gamma_{++}t}]a(0) + \tfrac{1}{2}[1 - \sigma_+ + (1 + \sigma_+)\,\mathrm{e}^{-2\Gamma_{++}t}]c(0), \tag{46}$$

$$d(t) = \tfrac{1}{2}(1 - \sigma_-)[1 - \mathrm{e}^{-2\Gamma_{--}t}]b(0) + \tfrac{1}{2}[1 - \sigma_- + (1 + \sigma_-)\,\mathrm{e}^{-2\Gamma_{--}t}]d(0), \tag{47}$$

$$x(t) = \mathrm{e}^{-\mathrm{i}(\omega_A + \omega_B + \Delta_{+-})t - \Gamma_{+-}t}x(0), \tag{48}$$

$$y(t) = \mathrm{e}^{-\mathrm{i}(\omega_A - \omega_B + \Delta_{-+})t - \Gamma_{-+}t}y(0). \tag{49}$$

The entanglement of the two-qubit state can be measured by the concurrence [39]. For the X-state $\hat{W}_Q(t)$, the concurrence is given by [35]

$$C(t) = 2\max[0, |x(t)| - \sqrt{b(t)c(t)}, |y(t)| - \sqrt{a(t)d(t)}], \tag{50}$$

which will be used to investigate the decay of the two-qubit entanglement.

5. Relaxation process of the single qubit

We investigate the relaxation process of qubit A interacting with the thermal reservoir, which can be controlled by the qubit B. We suppose that the qubit A is initially in a pure state $|\psi_A\rangle = \alpha|0\rangle + \beta|1\rangle$ ($|\alpha|^2 + |\beta|^2 = 1$) and qubit B is prepared in a quantum state described by a density matrix $\hat{\rho}_B$. The initial state of the two-qubit system is $\hat{W}_Q(0) = |\psi_A\rangle\langle\psi_A| \otimes \hat{\rho}_B$. In this case, the parameters in equations (34) and (35) are given respectively by $a_1(0) = P_0$, $a(0) = 2\alpha^*\beta P_0$, $a_z(0) = (|\alpha|^2 - |\beta|^2)P_0$, $b_1(0) = P_1$, $b(0) = 2\alpha^*\beta P_1$ and $b_z(0) = (|\alpha|^2 - |\beta|^2)P_1$ with $P_0 = \langle 0|\hat{\rho}_B|0\rangle$ and $P_1 = \langle 1|\hat{\rho}_B|1\rangle$. Then the reduced quantum state of qubit A is given by

$$\begin{aligned}\hat{\rho}_A(t) = \tfrac{1}{2} &+ \alpha^*\beta[\mathrm{e}^{-\mathrm{i}(\omega_A+J+\Delta_{++})t-\Gamma_{++}t}P_0 + \mathrm{e}^{-\mathrm{i}(\omega_A-J+\Delta_{--})t-\Gamma_{--}t}P_1]\hat{\sigma}^+ \\ &+ \alpha\beta^*[\mathrm{e}^{\mathrm{i}(\omega_A+J+\Delta_{++})t-\Gamma_{++}t}P_0 + \mathrm{e}^{\mathrm{i}(\omega_A-J+\Delta_{--})t-\Gamma_{--}t}P_1]\hat{\sigma}^- \\ &+ \tfrac{1}{2}[(|\alpha|^2 - |\beta|^2)(\mathrm{e}^{-2\Gamma_{++}t}P_0 + \mathrm{e}^{-2\Gamma_{--}t}P_1) \\ &+ \sigma_+(1-\mathrm{e}^{-2\Gamma_{++}t})P_0 + \sigma_-(1-\mathrm{e}^{-2\Gamma_{--}t})P_1]\hat{\sigma}_z. \end{aligned} \tag{51}$$

The average values of $\hat{\sigma}_z$ and $\hat{\sigma}_\pm$ are calculated to be

$$\left\langle\hat{\sigma}_A^z(t)\right\rangle = (|\alpha|^2 - |\beta|^2)(\mathrm{e}^{-2\Gamma_{++}t}P_0 + \mathrm{e}^{-2\Gamma_{--}t}P_1) + \sigma_+(1-\mathrm{e}^{-2\Gamma_{++}t})P_0 + \sigma_-(1-\mathrm{e}^{-2\Gamma_{--}t})P_1, \tag{52}$$

$$\langle\hat{\sigma}^+(t)\rangle = \alpha\beta^*[\mathrm{e}^{\mathrm{i}(\omega_A+J+\Delta_{++})t-\Gamma_{++}t}P_0 + \mathrm{e}^{\mathrm{i}(\omega_A-J+\Delta_{--})t-\Gamma_{--}t}P_1]. \tag{53}$$

This result shows that the relaxation times of $\left\langle\hat{\sigma}_A^z(t)\right\rangle$ and $\langle\hat{\sigma}^+(t)\rangle$ are $1/2\Gamma_{++}$ and $1/\Gamma_{++}$ for $P_0 = 1$ while they are $1/2\Gamma_{--}$ and $1/\Gamma_{--}$ for $P_1 = 1$. The equilibrium value of $\left\langle\hat{\sigma}_A^z(t)\right\rangle$ becomes

$$\left\langle\hat{\sigma}_A^z(\infty)\right\rangle = \sigma_+P_0 + \sigma_-P_1. \tag{54}$$

These results imply that the relaxation process of qubit A strongly depends on the state of qubit B. Since qubit B has the Ising-type coupling with qubit A and it does not interact with the thermal reservoir, the diagonal elements of the density matrix of qubit B remains unchanged during the time evolution. Thus, it is not remarkable that the population P_0 (or P_1) of the initial state of qubit B appears in the equilibrium value $\left\langle\hat{\sigma}_A^z(\infty)\right\rangle$ of qubit A. If we ignore the interaction effects on the relaxation process, we have $\left\langle\hat{\sigma}_A^z(t)\right\rangle = (|\alpha|^2 - |\beta|^2)\,\mathrm{e}^{-2\Gamma_0 t} + \sigma_0(1-\mathrm{e}^{-2\Gamma_0 t})$ and $\left\langle\hat{\sigma}_A^+(t)\right\rangle = \alpha\beta^*\,\mathrm{e}^{\mathrm{i}(\omega_A+\Delta_0)t-\Gamma_0 t}$ which do not depend on the state of the qubit B. In particular, when the thermal reservoir is in the vacuum state ($T = 0$), we obtain

$$\left\langle\hat{\sigma}_A^z(\infty)\right\rangle = \begin{cases} -1 & (0 < J < \omega_A), \\ 1 - 2P_0 & (0 < \omega_A < J), \\ 2P_0 - 1 & (0 < \omega_A < -J), \\ -1 & (0 < -J < \omega_A). \end{cases} \tag{55}$$

It is obvious that when $T = 0$, qubit A relaxes into the lower energy state. The parameters (ω_A, J) and qubit B determine which state $|0\rangle$ or $|1\rangle$ has the lower energy.

In order to investigate the relaxation process, we have to determine the damping constants Γ_{jk} and phase shifts Δ_{jk}. For this purpose, we assume the Ohmic dissipation [40] in which the spectral density of the qubit–reservoir interaction is given by

$$\sum_k g_k^2\delta(\omega-\omega_k) = \frac{1}{\pi}G\omega\,\mathrm{e}^{-\omega/\omega_c} \equiv D(\omega), \tag{56}$$

where G is a positive constant which characterizes the interaction strength and ω_c is a cut-off frequency of the thermal reservoir. Then, we can derive

$$\lambda^2 \int_0^\infty \mathrm{d}t \left\langle \left(\sum_k g_k \hat{a}_k \,\mathrm{e}^{-\mathrm{i}\omega_k t} \right) \left(\sum_k g_k \hat{a}_k^\dagger \right) \right\rangle \mathrm{e}^{\mathrm{i}\omega t} = \pi D(\omega)[\bar{n}(\omega) + 1] + \mathrm{i} \int_0^\infty \mathrm{d}\omega' \frac{\mathcal{P}}{\omega - \omega'} D(\omega')[\bar{n}(\omega') + 1]. \tag{57}$$

$$\lambda^2 \int_0^\infty \mathrm{d}t \left\langle \left(\sum_k g_k \hat{a}_k^\dagger \,\mathrm{e}^{\mathrm{i}\omega_k t} \right) \left(\sum_k g_k \hat{a}_k \right) \right\rangle \mathrm{e}^{\mathrm{i}\omega t} = \pi D(\omega)\bar{n}(\omega) + \mathrm{i} \int_0^\infty \mathrm{d}\omega' \frac{\mathcal{P}}{\omega - \omega'} D(\omega')\bar{n}(\omega'), \tag{58}$$

where $\bar{n}(\omega) = (\mathrm{e}^{\hbar\omega/k_\mathrm{B}T} - 1)^{-1}$ and $\mathcal{P}$ stands for the principal value integral. Then from the consideration in section 2, we obtain the damping parameters

$$\Gamma_{++} = G(\omega_A + J)\,\mathrm{e}^{-(\omega_A+J)/\omega_c}[2\bar{n}(\omega_A + J) + 1], \tag{59}$$

$$\Gamma_{--} = G(\omega_A - J)\,\mathrm{e}^{-(\omega_A-J)/\omega_c}[2\bar{n}(\omega_A - J) + 1], \tag{60}$$

$$\Gamma_{+-} = G(\omega_A + J)\,\mathrm{e}^{-(\omega_A+J)/\omega_c}[\bar{n}(\omega_A + J) + 1] + G(\omega_A - J)\,\mathrm{e}^{-(\omega_A-J)/\omega_c}\bar{n}(\omega_A - J), \tag{61}$$

$$\Gamma_{-+} = G(\omega_A - J)\,\mathrm{e}^{-(\omega_A-J)/\omega_c}[\bar{n}(\omega_A - J) + 1] + G(\omega_A + J)\,\mathrm{e}^{-(\omega_A+J)/\omega_c}\bar{n}(\omega_A + J), \tag{62}$$

for $\omega_A > |J|$ and

$$\Gamma_{++} = G(\omega_A + J)\,\mathrm{e}^{-(\omega_A+J)/\omega_c}[2\bar{n}(\omega_A + J) + 1], \tag{63}$$

$$\Gamma_{--} = G(|\omega_A - J|)\,\mathrm{e}^{-|\omega_A-J|/\omega_c}[2\bar{n}(|\omega_A - J|) + 1], \tag{64}$$

$$\Gamma_{+-} = G(\omega_A + J)\,\mathrm{e}^{-(\omega_A+J)/\omega_c}[\bar{n}(\omega_A + J) + 1] + G(|\omega_A - J|)\,\mathrm{e}^{-|\omega_A-J|/\omega_c}[\bar{n}(|\omega_A - J|) + 1], \tag{65}$$

$$\Gamma_{-+} = G(|\omega_A - J|)\,\mathrm{e}^{-|\omega_A-J|/\omega_c}\bar{n}(|\omega_A - J|) + G(\omega_A + J)\,\mathrm{e}^{-(\omega_A+J)/\omega_c}\bar{n}(\omega_A + J), \tag{66}$$

for $J > \omega_A$ and

$$\Gamma_{++} = G|\omega_A + J|\,\mathrm{e}^{-|\omega_A+J|/\omega_c}[2\bar{n}(|\omega_A + J|) + 1], \tag{67}$$

$$\Gamma_{--} = G(\omega_A - J)\,\mathrm{e}^{-(\omega_A-J)/\omega_c}[2\bar{n}(\omega_A - J) + 1], \tag{68}$$

$$\Gamma_{+-} = G|\omega_A + J|\,\mathrm{e}^{-|\omega_A+J|/\omega_c}\bar{n}(|\omega_A + J|) + G(\omega_A - J)\,\mathrm{e}^{-(\omega_A-J)/\omega_c}\bar{n}(\omega_A - J), \tag{69}$$

$$\Gamma_{-+} = G(\omega_A - J)\,\mathrm{e}^{-(\omega_A-J)/\omega_c}[\bar{n}(\omega_A - J) + 1] + G|\omega_A + J|\,\mathrm{e}^{-|\omega_A+J|/\omega_c}[\bar{n}(|\omega_A + J|) + 1], \tag{70}$$

for $-J > \omega_A$. In particular, when the thermal reservoir is in the vacuum state ($T = 0$), we find that $\lim_{T\to 0}\Gamma_{-+} = 0$ for $J > \omega_A$ and $\lim_{T\to 0}\Gamma_{+-} = 0$ for $-J > \omega_A$. If we ignore the effect of the interaction between the qubits on the relaxation process, the damping operator $\hat{\mathcal{L}}_Q^{(1)}$ vanishes and thus all the damping constants are equal to $\Gamma_0 = G\omega_A\,\mathrm{e}^{-\omega_A/\omega_c}[2\bar{n}(\omega_A) + 1]$.

In this paper, although we have assumed the Ohmic dissipation of the thermal reservoir, the phenomenological reservoir model [25, 41] can be also considered. In [25, 41], the functions

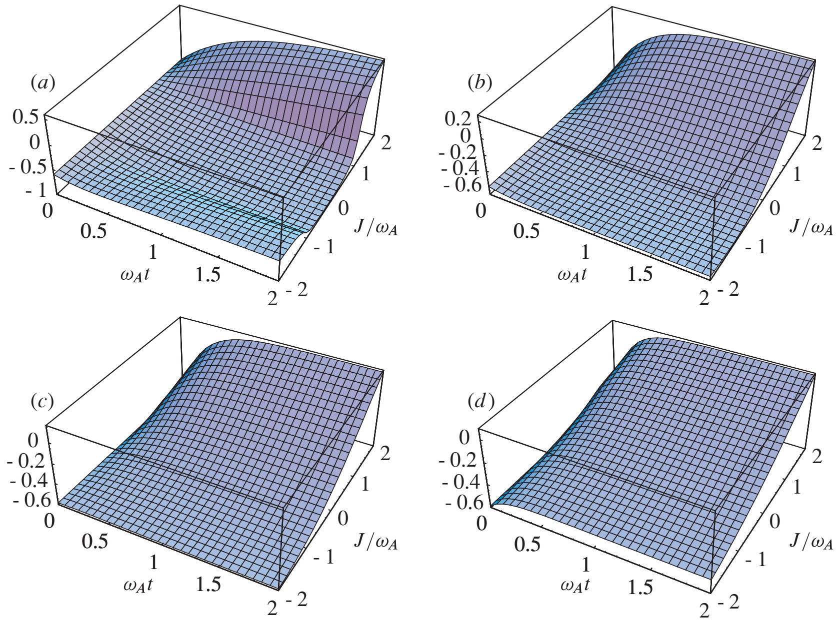

Figure 2. The time evolution of the average value $\langle\hat{\sigma}_A^z(t)\rangle$ in the cases of (*a*) $k_{\mathrm{B}}T/\hbar\omega_A = 0.01$, (*b*) $k_{\mathrm{B}}T/\hbar\omega_A = 0.8$, (*c*) $k_{\mathrm{B}}T/\hbar\omega_A = 1.2$ and (*d*) $k_{\mathrm{B}}T/\hbar\omega_A = 1.8$. In the figure, we set $G = 1.0$, $\omega_c = 5.0$, $P_0 = 0.2$ and $\langle\hat{\sigma}_A^z(0)\rangle = -0.6$.

$\phi_{-+}(\omega)$ and $\phi_{+-}(\omega)$, the real parts of which must satisfy the detailed balance condition, are assumed to be

$$\phi_{-+}(\omega_A) = G\frac{\mathrm{e}^{\hbar\omega_A/2k_{\mathrm{B}}T}}{\cosh\left(\frac{\hbar\omega_A}{2k_{\mathrm{B}}T}\right)}, \qquad \phi_{+-}(\omega_A) = G\frac{\mathrm{e}^{-\hbar\omega_A/2k_{\mathrm{B}}T}}{\cosh\left(\frac{\hbar\omega_A}{2k_{\mathrm{B}}T}\right)}. \tag{71}$$

Then the damping parameters are given by

$$\Gamma_{++} = \Gamma_{--} = 2G, \tag{72}$$

$$\Gamma_{+-} = G\frac{\mathrm{e}^{\hbar(\omega_A+J)/2k_{\mathrm{B}}T}}{\cosh\left[\frac{\hbar(\omega_A+J)}{2k_{\mathrm{B}}T}\right]} + G\frac{\mathrm{e}^{-\hbar(\omega_A-J)/2k_{\mathrm{B}}T}}{\cosh\left[\frac{\hbar(\omega_A-J)}{2k_{\mathrm{B}}T}\right]}, \tag{73}$$

$$\Gamma_{-+} = G\frac{\mathrm{e}^{\hbar(\omega_A-J)/2k_{\mathrm{B}}T}}{\cosh\left[\frac{\hbar(\omega_A-J)}{2k_{\mathrm{B}}T}\right]} + G\frac{\mathrm{e}^{-\hbar(\omega_A+J)/2k_{\mathrm{B}}T}}{\cosh\left[\frac{\hbar(\omega_A+J)}{2k_{\mathrm{B}}T}\right]}, \tag{74}$$

which satisfy $\lim_{T\to 0}\Gamma_{-+} = 0$ for $J > \omega_A$ and $\lim_{T\to 0}\Gamma_{+-} = 0$ for $-J > \omega_A$. All the phase shifts Δ_{++}, Δ_{--}, Δ_{+-}, Δ_{-+} vanish in the phenomenological model. The Ohmic dissipation model and the phenomenological model yield qualitatively the same results in our consideration.

We now investigate the relaxation process of qubit A. Since the coherence given by $\left\langle\hat{\sigma}_A^+(t)\right\rangle$ decays to zero exponentially with the two damping constants Γ_{++} and Γ_{--}, we confine ourselves to considering the population difference $\left\langle\hat{\sigma}_A^z(t)\right\rangle$ between the two states of qubit A. The time evolution is plotted in figure 2.

We find from the figure that the time evolution of the average value strongly depends on the parameters J/ω_A and P_0 in the low temperature region ($k_{\mathrm{B}}T/\hbar\omega_A \ll 1$) while the

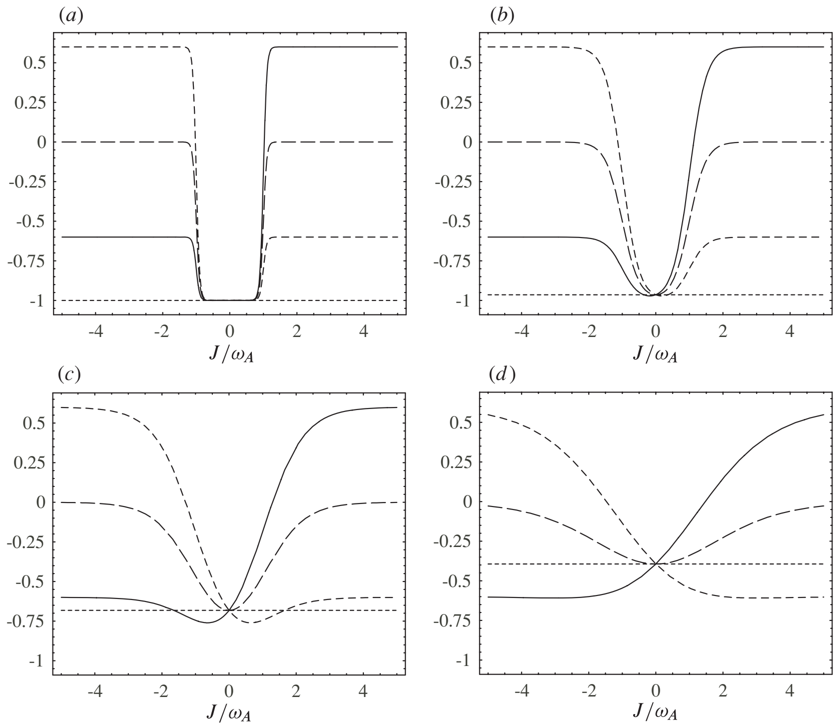

Figure 3. The dependence of the equilibrium value $\langle\hat{\sigma}_A^z(\infty)\rangle$ on the ratio J/ω_A in the cases of (*a*) $k_BT/\hbar\omega_A = 0.05$, (*b*) $k_BT/\hbar\omega_A = 0.25$, (*c*) $k_BT/\hbar\omega_A = 0.6$ and (*d*) $k_BT/\hbar\omega_A = 1.2$. The solid line is for $P_0 = 0.2$, the long-dashed line for $P_0 = 0.5$ and the short-dashed line for $P_0 = 0.8$. The dotted horizontal line represents the equilibrium value that is obtained when the interaction effects on the relaxation process are ignored.

dependence becomes weak in the high temperature region ($k_BT/\hbar\omega_A \gg 1$). The equilibrium value of $\langle\hat{\sigma}_A(t)\rangle$, namely $\left\langle\hat{\sigma}_A^z(\infty)\right\rangle$, is plotted in figure 3.

We see from this figure that the equilibrium value can be controlled by setting the parameters J/ω_A and P_0 in the low temperature region.

6. Decay of two-qubit entanglement

In this section, we investigate the decay of entanglement of the two qubits which are initially prepared in the *X*-state given by equation (42). In the time evolution governed by equation (13), the *X*-state is preserved and the two qubits at time *t* are still in the *X*-state (43). Then the entanglement of the two qubits can be calculated from equation (50).

We first consider the case that the thermal reservoir is in the vacuum state ($T = 0$). When the equality $J > \omega_A$ holds, we obtain the matrix elements of the *X*-state from equations (44)–(49):

$$a(t) = e^{-2\Gamma_{++}t}a(0), \tag{75}$$

$$b(t) = b(0) + (1 - e^{-2\Gamma_{--}t})d(0), \tag{76}$$

$$c(t) = (1 - e^{-2\Gamma_{++}t})a(0) + c(0), \tag{77}$$

$$d(t) = e^{-2\Gamma_{--}t} d(0), \tag{78}$$

$$x(t) = e^{-i(\omega_A+\omega_B+\Delta_{+-})t - \Gamma_{+-}t} x(0), \tag{79}$$

$$y(t) = e^{-i(\omega_A-\omega_B+\Delta_{-+})t} y(0), \tag{80}$$

where we have used the fact that $\Gamma_{-+} = 0$ if $J > \omega_A$ and $T = 0$. When $x(0) \neq 0$ and $y(0) = 0$, the entanglement sudden-death (ESD) [42, 43] takes place if the initial state is entangled. On the other hand, when $x(0) = 0$ and $y(0) \neq 0$, we obtain the concurrence

$$C(t) = 2\max[0, |y(0)| - e^{-(\Gamma_{++}+\Gamma_{--})t}\sqrt{a(0)d(0)}]. \tag{81}$$

This result shows that the entanglement can be created at the time t_e:

$$t_e = \frac{1}{\Gamma_{++}+\Gamma_{--}} \ln\left(\frac{\sqrt{a(0)d(0)}}{|y(0)|}\right), \tag{82}$$

even though there is no entanglement in the initial state, that is, $\sqrt{a(0)d(0)} \geqslant |y(0)|$. This means that the entanglement sudden-birth (ESB) [44] occurs at the time t_e. In the equilibrium state, we have $C(\infty) = 2|y(0)|$. Therefore, we have found that the two qubits initially prepared in the X-state with $y(0) \neq 0$ are always entangled in the equilibrium state.

In the case that the inequality $-J > \omega_A$ holds, we obtain from equations (44)–(49)

$$a(t) = a(0) + (1 - e^{-2\Gamma_{++}t})c(0), \tag{83}$$

$$b(t) = e^{-2\Gamma_{--}t} b(0), \tag{84}$$

$$c(t) = e^{-2\Gamma_{++}t} c(0), \tag{85}$$

$$d(t) = (1 - e^{-2\Gamma_{--}t})b(0) + d(0), \tag{86}$$

$$x(t) = e^{-i(\omega_A+\omega_B+\Delta_{+-})t} x(0), \tag{87}$$

$$y(t) = e^{-i(\omega_A-\omega_B+\Delta_{-+})t - \Gamma_{-+}t} y(0), \tag{88}$$

where we have used the fact that $\Gamma_{+-} = 0$ when $J > \omega_A$ and $T = 0$. The ESD takes place when $x(0) = 0$ and $y(0) \neq 0$ while the ESB occurs when $x(0) \neq 0$ and $y(0) = 0$. In the latter case, the concurrence is given by

$$C(t) = 2\max[0, |x(0)| - e^{-(\Gamma_{++}+\Gamma_{--})t}\sqrt{b(0)c(0)}], \tag{89}$$

which implies that although there is no entanglement in the initial state with $\sqrt{a(0)d(0)} \geqslant |y(0)|$, the entanglement is created at the time t_e:

$$t_e = \frac{1}{\Gamma_{++}+\Gamma_{--}} \ln\left(\frac{\sqrt{b(0)c(0)}}{|x(0)|}\right). \tag{90}$$

In the equilibrium state, we have $C(\infty) = 2|x(0)|$. Therefore, the two qubits initially prepared in the X-state with $x(0) \neq 0$ are always entangled in the equilibrium state.

On the other hand, when the inequality $\omega_A > |J|$ is satisfied, we obtain

$$a(t) = e^{-2\Gamma_{+}t} a(0), \tag{91}$$

$$b(t) = e^{-2\Gamma_{-}t} b(0), \tag{92}$$

$$c(t) = (1 - e^{-2\Gamma_{+}t})a(0) + c(0), \tag{93}$$

$$d(t) = (1 - e^{-2\Gamma_{-}t})b(0) + d(0), \tag{94}$$

$$x(t) = e^{-i(\omega_A+\omega_B+\Delta_{+-})t - \Gamma_{+}t} x(0), \tag{95}$$

Table 2. The lower energy state of qubit A.

The parameters	The state of B	The lower energy state of A
$\omega_A > \lvert J \rvert$	$\lvert 0\rangle$ or $\lvert 1\rangle$	$\lvert 1\rangle$
$-J > \omega_A > J$	$\lvert 0\rangle$ ($\lvert 1\rangle$)	$\lvert 0\rangle$ ($\lvert 1\rangle$)
$J > \omega_A > -J$	$\lvert 0\rangle$ ($\lvert 1\rangle$)	$\lvert 1\rangle$ ($\lvert 0\rangle$)

$$y(t) = \mathrm{e}^{-\mathrm{i}(\omega_A - \omega_B + \Delta_{+-})t - \Gamma_{-}t} y(0), \tag{96}$$

which yields the concurrence,

$$C(t) = 2\max[0, C_1(t), C_2(t)] \tag{97}$$

with

$$C_1(t) = |y(0)|\,\mathrm{e}^{-\Gamma_{-+}t} - \mathrm{e}^{-\Gamma_{++}}\sqrt{a(0)[(1-\mathrm{e}^{-2\Gamma_{-}t})b(0) + d(0)]}, \tag{98}$$

$$C_2(t) = |x(0)|\,\mathrm{e}^{-\Gamma_{+-}t} - \mathrm{e}^{-\Gamma_{--}}\sqrt{b(0)[(1-\mathrm{e}^{-2\Gamma_{+}t})a(0) + c(0)]}. \tag{99}$$

Thus for any initial X-state, there is no entanglement in the equilibrium state.

In the absence of the thermal reservoir, the Ising-type interaction between the two qubits cannot create the entanglement. The concurrence remains unchanged during the time evolution of the two qubits with the Ising-type interaction. In fact, we have $a(t) = a(0)$, $b(t) = b(0)$, $c(t) = c(0)$, $d(t) = d(0)$, $x(t) = x(0)\,\mathrm{e}^{-\mathrm{i}(\omega_A+\omega_B)t}$ and $y(t) = y(0)\,\mathrm{e}^{-\mathrm{i}(\omega_A-\omega_B)t}$ for the X-state. It is also obvious that the thermal reservoir which locally interacts with the qubit destructs the entanglement and does not create it. Therefore, the ESB is caused by the synthetic effect of the qubit–qubit interaction and the qubit–reservoir interaction. Mathematically the damping operator $\hat{\mathcal{L}}_Q^{(1)}$ including the interaction effect has the essential importance in the ESB. It is obvious that qubit A relaxes into the lower energy state under the influence of the thermal reservoir with $T = 0$. The lower energy state of qubit A interacting with qubit B is determined by the parameters ω_A and J and the state of qubit B (see table 2).

It is understood from the table that the entanglement can exist in the equilibrium state when $\omega_A < |J|$ and qubit B is in a superposition of $|0\rangle$ and $|1\rangle$.

We next consider the decay of the entanglement under the influence of the thermal reservoir with a finite temperature ($T \neq 0$). For this purpose, we suppose that the two qubits are initially prepared in the Werner state [36]:

$$\hat{W}_F(0) = F|\Psi_+\rangle\langle\Psi_+| + \frac{1-F}{3}(\hat{1}\otimes\hat{1} - |\Psi_+\rangle\langle\Psi_+|), \tag{100}$$

with $|\Psi_+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$, which is entangled if and only if $F > 1/2$ since the concurrence is given by $C(0) = \max[2F-1, 0]$. Substituting $a(0) = d(0) = \frac{1}{3}(1-F)$ $b(0) = c(0) = \frac{1}{6}(1+2F)$, $x(0) = 0$ and $y(0) = \frac{1}{6}(4F-1)$ into equations (44)–(49), we obtain the Werner state $\hat{W}_F(t)$ at time t. The time evolution of the concurrence of the Werner state $\hat{W}_F(t)$ is plotted in figure 4. We find from the figure that the ESB takes place when J/ω_J is large. In the case of the thermal reservoir with a finite temperature, the entanglement created by the ESB decays with time.

Finally we consider the violation of the Bell inequality [45] in the two-qubit state $\hat{W}_Q(t)$. For this purpose, we introduce a 3×3 matrix $\mathsf{T}(t)$, whose matrix element is defined by $\mathsf{T}_{\mu\nu}(t) = \mathrm{Tr}[(\hat{\sigma}^{\mu}\otimes\hat{\sigma}^{\nu})\hat{W}_Q(t)]$. We further introduce a symmetric matrix $\mathsf{U}(t) = \mathsf{T}^{\mathrm{T}}(t)\mathsf{T}(t)$. Here we denote the eigenvalues of $\mathsf{U}(t)$ as $u_1(t)$, $u_2(t)$ and $u_3(t)$ with decreasing order.

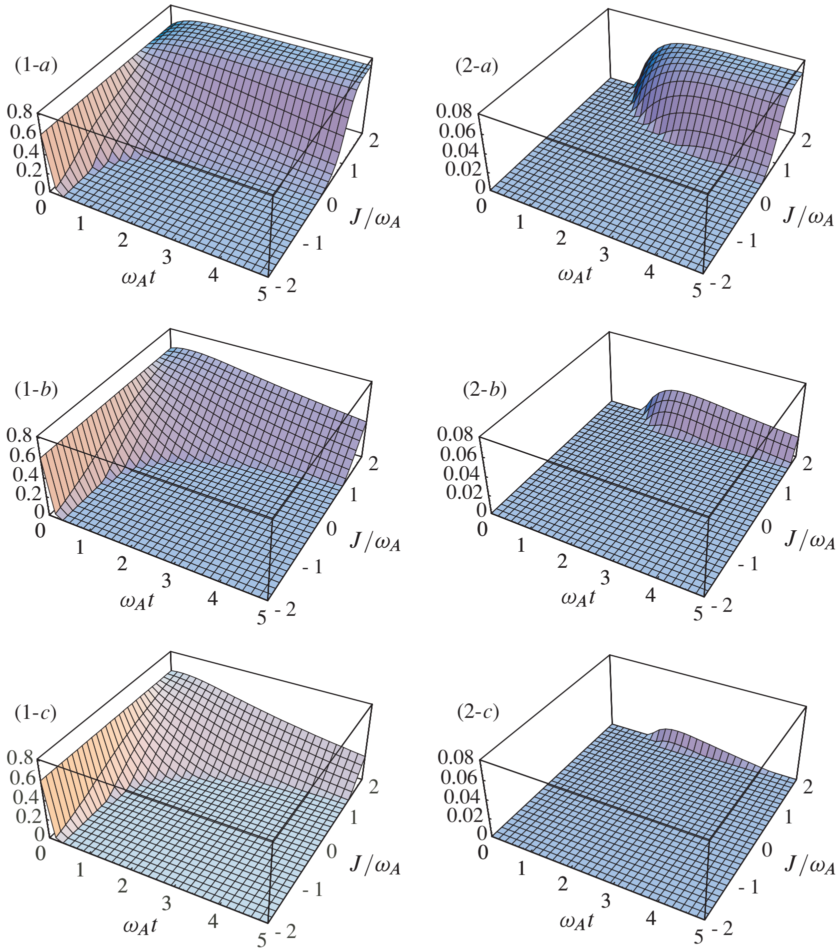

Figure 4. The concurrence of the Werner state $\hat{W}_F(t)$ in the cases of $F = 0.8$ [(1-*a*), (1-*b*) and (1-*c*)] and $F = 0.3$ ((2-*a*), (2-*b*) and (2-*c*)), where we set $k_B T/\hbar\omega_A = 0.1$ for (1-*a*) and (2-*a*) and $k_B T/\hbar\omega_A = 0.48$ for (2-*a*) and (2-*a*) and $k_B T/\hbar\omega_A = 0.58$ for (3-*a*) and (3-a). In all the figures, $G = 1.0$ and $\omega_c/\omega_A = 5.0$ are assumed.

Then the maximal average value of the Bell operator is given by $\langle\hat{\mathcal{B}}(t)\rangle = 2\sqrt{u_1(t) + u_2(t)}$ [46]. If the inequality $\langle\hat{\mathcal{B}}(t)\rangle > 2$ is satisfied, the Bell inequality is violated in the two-qubit state $\hat{W}_Q(t)$. Thus we can quantify the Bell inequality violation by $B(t) = \max[u_1(t) + u_2(t) - 1, 0]$. The Bell inequality is satisfied if and only if $B(t) = 0$. In the case of the thermal reservoir with $T = 0$, we obtain $B(\infty) = \max[0, 4|y(0)|^2]$ for $J > \omega_A$ and we obtain $B(\infty) = \max[0, 4|x(0)|^2]$ for $-J > \omega_A$. Therefore, we find that if $y(0) \neq 0$ and $J > \omega_A$ or if $x(0) \neq 0$ and $-J > \omega_A$, the Bell inequality is violated in the equilibrium state, even though it is fulfilled in the initial *X*-state.

7. Concluding remarks

In this paper, we have investigated the relaxation process of the interacting two-qubit system, where one of the two qubits interacts with the thermal reservoir and the other is isolated from its environmental system. Using the projection operator method, we have derived the quantum master equation of the two-qubit system. The damping operator consists of two parts; one includes the effect of the interaction between the qubits and other does not. The interaction effects have essential importance in the relaxation process of the two-qubit system. In fact, the relaxation times and equilibrium value of the qubits strongly depend on the interaction and the initial state. If we ignore the interaction effect, the relaxation process of the qubit does not depend on them. Furthermore we have investigated the decay of the two-qubit entanglement. We have found that not only the entanglement sudden-death but also the entanglement sudden-birth can take place. Although the former is caused by the interaction with the thermal reservoir, the latter is due to the synthetic effect of the qubit–qubit interaction and the qubit–reservoir interaction. The synthetic effect can also violate the Bell inequality, even if it is satisfied in the initial state. Finally it is important to note that we have assumed the rotating-wave approximation for the interaction between the qubit and the thermal reservoir, and thus some restrictions are applied to our results. For example, since the rotating-wave approximation is broken down if $\omega_A \approx |J|$ is fulfilled, our results may be modified in this case and so we need a further investigation. We will study the relaxation process of the interacting qubits beyond the rotating-wave approximation in a future work. In this paper we have investigated the decay of the single-particle coherence of qubit A which directly interacts with the thermal reservoir and qubit B. We have also studied the decay of the two-qubit entanglement. The decoherence of qubit B which indirectly interacts with the thermal reservoir is also important. This problem has already been studied in [47, 48].

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